Solution Dimensions of Matrix Differential Equations

Andrew K. Greene   Kevin Garcia   Dorian Persaud

June 30, 2017

Abstract

This paper studies the possible dimensions of solution spaces for first-order matrix differential equations over $M_2(\mathbb{C})$. MDEs are purely algebraic, noncommutative analogues of classical ordinary differential equations in which functions are replaced by matrices and differentiation is replaced by a derivation. An elementary proof is provided that shows all derivations on $M_2(\mathbb{C})$ are inner. A coefficient matrix is derived that encodes key features of the MDE. In particular, Gaussian elimination is used to determine which solution dimensions are possible and impossible. However, the coefficient matrix has variable entries, so a game-like, case-based analysis is carried out. An eigenvalue approach is also offered as an alternative proof.

1 Introduction

In this paper we determine the possible dimensions of solution spaces for first-order matrix differential equations over $M_2(\mathbb{C})$. MDEs in this paper are purely algebraic, noncommutative analogues of classical ordinary differential equations (ODEs) in which functions are replaced by matrices and differentiation is replaced by a derivation to be defined below. We show how an MDE over $M_n(\mathbb{C})$ reduces to a linear system of n^2 equations in n^2 unknowns. We must warn the reader that our terminology is nonstandard. Elsewhere, a matrix differential equation more commonly refers to a differential equation involving matrix-valued and vector-valued functions. An example of what is commonly referred to as a matrix differential equation is

$$\frac{d}{dt}\vec{x}(t) = A(t)\vec{x}(t)$$

for some matrix-valued $A(t)$ and vector-valued $\vec{x}(t)$ functions defined on an interval $I \subset \mathbb{R}$. In the present work, there is no independent variable t, no functions of t, and consequently, no differentiation. Briefly, an MDE is an analogue a classical ODE with matrices replacing functions and a derivation Δ replacing differentiation. To illustrate, a classical first-order ODE with constant coefficients may be written in the form

$$y'(t) + ay(t) = g(t).$$

The analogous MDE is

$$\Delta(Y) + aY = G$$

where $Y \in M_n(\mathbb{C})$ is the unknown matrix, $\Delta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map satisfying the product rule, $a \in \mathbb{C}$ is a scalar, and $G \in M_n(\mathbb{C})$ is a given matrix (the nonhomogeneous term). The MDE analogue of the variable-coefficient ODE

$$y'(t) + a(t)y(t) = g(t)$$

is

$$\Delta(Y) + AY = G$$

where A is now a given matrix A instead of a scalar.

A purely algebraic approach to differential equations was initiated in 1930s by Ritt, cf. [Rit32]. However, the focus of differential algebra has traditionally been on commutative differential algebras, although there has been work more recently on noncommutative differential algebras, cf. [GL14]. Although general context of this article is noncommutative differential algebra, all results in this paper are elementary and focus on $M_2(\mathbb{C})$, the algebra of 2×2 matrices with complex entries.

Including this introduction, this article has five sections as well as an appendix. In section 2, we offer an elementary proof of the fact that all derivations on $M_2(\mathbb{C})$ are inner, i.e., defined by commutators. This classic result dates back to the 1930s in the work of Wedderburn, Noether, Jacobson, and Hochschild on semisimple Lie algebras and semisimple aglebras, cf. [Hoc42; Jac37; Noe33; Wed08]. The proof given below uses elementary matrix algebra and the product rule. Consequently, MDEs are equivalent to algebraic matrix equations and can be solved and analyzed using linear algebra. Section 3 computes the coefficient matrix corresponding to an MDE. In contrast to classical ODEs, in which the order of a homogeneous equation matches its solution dimension, cf. [BD69], MDEs can have a variety of solution dimensions. In particular, the dimension of solutions spaces for first-order homogeneous MDEs over $M_2(\mathbb{C})$ can be any integer between 0 and 4, but curiously not 3. We give examples in Section 4 to demonstrate the possible solution dimensions. In section 5 we provide two proofs that dimension 3 is impossible. One proof applies Gaussian elimination to a coefficient matrix of variables. The other proof uses the eigenvalues and eigenvectors of X. The choice of $\mathbb{C}$ as the field is not critical until eigenvalues are used in an alternate proof found at the end of the paper.

Definition 1.1. A map $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is a *derivation* if it is linear and satisfies the product rule

$$\Delta(Y_1, Y_2) = \Delta(Y_1)Y_2 + Y_1\Delta(Y_2) \text{ for all } Y_1, Y_2 \in M_2(\mathbb{C}).$$

A derivation Δ is *inner* if there exists $X \in M_2(\mathbb{C})$ with

$$\Delta(Y) = [Y, X] = YX - XY \text{ for all } Y \in M_2(\mathbb{C}).$$

In this case, we denote the inner derivation $\Delta = \Delta_X$.

2 Derivations on $M_2(\mathbb{C})$

In this section we prove that derivations on $M_2(\mathbb{C})$ are inner. This well-known fact is true more generally in $M_n(\mathbb{C})$ and is typically proved as a result about semisimple Lie algebras. We provide an elementary proof using matrix algebra and the product rule. To summarize, we write $\Delta(E_{ij})$ in terms of matrix units E_{ij}, repeatedly apply the product rule to various $E_{ij}E_{rs}$, and eventually are able to solve $\Delta = \Delta_X$ for a matrix X.

Proposition 2.1. Every derivation Δ on $M_2(\mathbb{C})$ is inner.

Proof. Every matrix in $M_2(\mathbb{C})$ is a unique linear combination of the linearly independent matrices

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, for each $1 \leq i, j \leq 2$,

$$\Delta(E_{ij}) = \sum_{r=1}^{2} \sum_{t=1}^{2} \lambda_{r,t,i,j} E_{rt} = \begin{pmatrix} \lambda_{1,1,i,j} & \lambda_{1,2,i,j} \\ \lambda_{2,1,i,j} & \lambda_{2,2,i,j} \end{pmatrix}$$

for some scalars $\lambda_{r,t,i,j} \in \mathbb{C}$. In total we have 16 scalars $\lambda_{r,s,i,j}$, but we can discover relations between them by applying the product rule of Δ to the following multiplication identities:

$$E_{ij}E_{st} = \delta_{js}E_{it}$$

where δ_{js} is the dirac delta function

$$\delta_{js} = \begin{cases} 1 & \text{if } j = s \\ 0 & \text{if } j \neq s \end{cases}.$$

Since $E_{11}^2 = E_{11}$,

$$\begin{aligned}
\begin{pmatrix} \lambda_{1,1,1,1} & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & \lambda_{2,2,1,1} \end{pmatrix} &= \Delta(E_{11}) \\
&= \Delta(E_{11}E_{11}) \\
&= \Delta(E_{11})E_{11} + E_{11}\Delta(E_{11}) \\
&= \begin{pmatrix} \lambda_{1,1,1,1} & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & \lambda_{2,2,1,1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1,1,1,1} & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & \lambda_{2,2,1,1} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_{1,1,1,1} & 0 \\ \lambda_{2,1,1,1} & 0 \end{pmatrix} + \begin{pmatrix} \lambda_{1,1,1,1} & \lambda_{1,2,1,1} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2\lambda_{1,1,1,1} & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & 0 \end{pmatrix}.
\end{aligned}$$

Consequently,

$$\Delta(E_{11}) = \begin{pmatrix} 0 & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & 0 \end{pmatrix}.$$

Similarly, $E_{22}^2 = E_{22}$ implies

$$\begin{aligned}
\begin{pmatrix} \lambda_{1,1,2,2} & \lambda_{1,2,2,2} \\ \lambda_{2,1,2,2} & \lambda_{2,2,2,2} \end{pmatrix} &= \Delta(E_{22}) \\
&= \Delta(E_{22}E_{22}) \\
&= \Delta(E_{22})E_{22} + E_{22}\Delta(E_{22}) \\
&= \begin{pmatrix} \lambda_{1,1,2,2} & \lambda_{1,2,2,2} \\ \lambda_{2,1,2,2} & \lambda_{2,2,2,2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{1,1,2,2} & \lambda_{1,2,2,2} \\ \lambda_{2,1,2,2} & \lambda_{2,2,2,2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \lambda_{1,2,2,2} \\ 0 & \lambda_{2,2,2,2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda_{2,1,2,2} & \lambda_{2,2,2,2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \lambda_{1,2,2,2} \\ \lambda_{2,1,2,2} & 2\lambda_{2,2,2,2} \end{pmatrix}
\end{aligned}$$

Consequently,

$$\Delta(E_{22}) = \begin{pmatrix} 0 & \lambda_{1,2,2,2} \\ \lambda_{2,1,2,2} & 0 \end{pmatrix}.$$

We can relate $\Delta(E_{11})$ and $\Delta(E_{22})$ by observing

$$\begin{aligned}
\Delta(E_{11}) + \Delta(E_{22}) &= \Delta(E_{11} + E_{22}) \\
&= \Delta(I) \\
&= \Delta(I^2) \\
&= \Delta(I)I + I\Delta(I) \\
&= 2\Delta(I).
\end{aligned}$$

The only way $\Delta(I) = 2\Delta(I)$ is if $\Delta(I) = 0$. Therefore,

$$\Delta(E_{11}) = -\Delta(E_{22}) = \begin{pmatrix} 0 & \lambda_{1,2,1,1} \\ \lambda_{2,1,1,1} & 0 \end{pmatrix}.$$

To simplify notation, rewrite

$$\Delta(E_{11}) = -\Delta(E_{22}) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Using $E_{12} = E_{11}E_{12}$,

$$\begin{aligned}
\begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ \lambda_{2,1,1,2} & \lambda_{2,2,1,2} \end{pmatrix} &= \Delta(E_{12}) \\
&= \Delta(E_{11}E_{12}) \\
&= \Delta(E_{11})E_{12} + E_{11}\Delta(E_{12}) \\
&= \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ \lambda_{2,1,1,2} & \lambda_{2,2,1,2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix}
\end{aligned}$$

Consequently,

$$\Delta(E_{12}) = \begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix}.$$

On the other hand, $E_{12} = E_{12}E_{22}$ implies

$$\begin{aligned}
\begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix} &= \Delta(E_{12})E_{22} + E_{12}\Delta(E_{22}) \\
&= \begin{pmatrix} \lambda_{1,1,1,2} & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix} + \begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -b & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix}
\end{aligned}$$

Therefore,

$$\Delta(E_{12}) = \begin{pmatrix} -b & \lambda_{1,2,1,2} \\ 0 & b \end{pmatrix} = \begin{pmatrix} -b & c \\ 0 & b \end{pmatrix}$$

where we have introduced c for convenience.

We argue similarly $E_{21} = E_{22}E_{21}$ and $E_{21} = E_{21}E_{11}$ to obtain

$$\begin{aligned}
\begin{pmatrix} \lambda_{1,1,2,1} & \lambda_{1,2,2,1} \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix} &= \Delta(E_{22})E_{21} + E_{22}\Delta(E_{21}) \\
&= \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_{1,1,2,1} & \lambda_{1,2,2,1} \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix} \\
&= \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix} \\
&= \begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix} = \Delta(E_{21})E_{11} + E_{21}\Delta(E_{11})$$
$$= \begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & \lambda_{2,2,2,1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$$
$$= \begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & a \end{pmatrix}.$$

Therefore,

$$\Delta(E_{21}) = \begin{pmatrix} -a & 0 \\ \lambda_{2,1,2,1} & a \end{pmatrix} = \begin{pmatrix} -a & 0 \\ d & a \end{pmatrix}$$

where we have introduced d for convenience.

Observe that $E_{12}E_{21} = E_{11}$. It follows that

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = \Delta(E_{12})E_{21} + E_{12}\Delta(E_{21})$$
$$= \begin{pmatrix} -b & c \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -a & 0 \\ d & a \end{pmatrix}$$
$$= \begin{pmatrix} c & 0 \\ b & 0 \end{pmatrix} + \begin{pmatrix} d & a \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} c+d & a \\ b & 0 \end{pmatrix}$$

We conclude that $d = -c$.

Combining our results for each of the four $\Delta(E_{ij})$, we have for arbitrary $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in M_2(\mathbb{C})$,

$$\Delta(Y) + y_{11}\Delta(E_{11}) + y_{12}\Delta(E_{12}) + y_{21}\Delta(E_{21}) + y_{22}\Delta(E_{22})$$
$$= y_{11}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} + y_{12}\begin{pmatrix} -b & c \\ 0 & b \end{pmatrix} + y_{21}\begin{pmatrix} -a & 0 \\ -c & a \end{pmatrix} + y_{22}\begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -by_{12} - ay_{21} & ay_{11} + cy_{12} - ay_{22} \\ by_{11} - cy_{21} - by_{22} & by_{12} + ay_{21} \end{pmatrix}.$$

To complete the proof, we will obtain the same matrix by expanding $\Delta_X(Y) = YX - XY$ with

$$X = \begin{pmatrix} -c & a \\ -b & 0 \end{pmatrix}.$$

We have

$$YX - XY = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} -c & a \\ -b & 0 \end{pmatrix} - \begin{pmatrix} -c & a \\ -b & 0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$
$$= \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} -c & a \\ -b & 0 \end{pmatrix} + \begin{pmatrix} c & -a \\ b & -0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$
$$= \begin{pmatrix} -cy_{11} - by_{12} & ay_{11} \\ -cy_{21} - by_{22} & ay_{21} \end{pmatrix} + \begin{pmatrix} cy_{11} - ay_{21} & cy_{12} - ay_{22} \\ by_{11} & by_{12} \end{pmatrix}$$
$$= \begin{pmatrix} -by_{12} - ay_{21} & ay_{11} + cy_{12} - ay_{22} \\ by_{11} - cy_{21} - by_{22} & by_{12} + ay_{21} \end{pmatrix}.$$

□

Remark 2.2. Note that at the end of the proof, we chose $X = \begin{pmatrix} -c & a \\ -b & 0 \end{pmatrix}$, but other choices of X will work as well provided that $x_{22} - x_{11} = c$. The one degree of freedom is a result of the fact that for any scalar λ, X and $X + \lambda I$ produce the same inner derivation:

$$[Y, X + kI] = [Y, X] + [Y, kI] = [Y, X] + 0.$$

3 Matrix Differential Equations

A consequence of derivations on $M_2(\mathbb{C})$ being inner is that matrix differential equations are algebraic equations. To see the type of algebraic equations that arise, fix $X \in M_2(\mathbb{C})$ and let Δ_X be the inner derivation defined by commutation with X. If $Y \in M_2(\mathbb{C})$, then the first-order constant-coefficient MDE

$$\Delta_X(Y) + aY = 0$$

is equivalent to

$$YX - XY + aY = 0.$$

Likewise, the second-order MDE

$$\Delta_X^2(Y) + a\Delta_X(Y) + bY = 0$$

is equivalent to

$$\begin{aligned} 0 &= \Delta(YX - XY) + a(YX - XY) + bY \\ &= (YX - XY)X - X(YX - XY) + a(YX - XY) + bY \\ &= YX^2 - 2XYX + X^2Y + aYX - aXY + bY. \end{aligned}$$

In the classical theory of differential equations, solutions are sought of the form e^{rx} by finding the roots of an auxillary polynomial that has the same coefficients as the ODE. In contrast, MDEs can be solved using linear algebra and solving a system of linear equations.

If the coefficients and entries of X were all given, then we could multiply out all matrix products, simplify, and view these equations as systems of four *linear* equations in four unknown variables $y_{11}, y_{12}, y_{21}, y_{22}$. The results in this paper focus exclusively on first-order equations, but the second-order example is provided above to demonstrate how these methods could also be applied for MDEs of higher order or for matrix algebras $M_n(\mathbb{C})$ for larger n. The resulting system has n^2 linear equations and n^2 unknowns. Thus, the size of the linear system depends only on the size of the matrices and not the differential order of the MDE. We work only with $n = 2$ in this paper because hands-only computation becomes too cumbersome otherwise. For larger values of n, a computer algebra system is recommended. Many insights for this paper were gained by using Maple and Sage.

The most general homogeneous, first-order MDE with constant coefficients has the form

$$c_1\Delta_X(Y) + c_0Y = 0.$$

By dividing by c_1, we may assume the following form which has one fewer parameters:

$$\Delta_X(Y) + aY = 0. \tag{3.1}$$

Using the definition of Δ_X, equation (3.1) can be rewritten as the matrix equation

$$XY - YX + aY = 0. \tag{3.2}$$

In terms of the entries of X and Y,

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} + a \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, after matrix multiplication and simplification, we obtain

$$\begin{pmatrix} ay_{11}+x_{21}y_{12}-x_{12}y_{21} & x_{12}y_{11}+ay_{12}-x_{11}y_{12}+x_{22}y_{12}-x_{12}y_{22} \\ -x_{21}y_{11}+ay_{21}+x_{11}y_{21}-x_{22}y_{21}+x_{21}y_{22} & -x_{21}y_{12}+x_{12}y_{21}+ay_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Each entries yields a separate equation, so we obtain the following linear system of four equations and four unknowns. The y_{ij} are the unknown variables. The a and x_{ij} are given constants.

$$\begin{aligned} ay_{11}+x_{21}y_{12}-x_{12}y_{21} &= 0 \\ x_{12}y_{11}+ay_{12}-x_{11}y_{12}+x_{22}y_{12}-x_{12}y_{22} &= 0 \\ -x_{21}y_{11}+ay_{21}+x_{11}y_{21}-x_{22}y_{21}+x_{21}y_{22} &= 0 \\ -x_{21}y_{12}+x_{12}y_{21}+ay_{22} &= 0. \end{aligned}$$

The equivalent matrix equation is

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a-x_{11}+x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a+x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.3}$$

Note that applying the above steps to an n-th order MDE will result in a matrix whose entries are polynomials in the x_{ij} variables and the coefficients of the MDE. Specifically, they will have degree at most n if the x_{ij} are treated as variables and the coefficients are treated as parameters. If the coefficients are also treated as variables, then the polynomials will have degree at most $n+1$. This could be useful to keep in mind if one's goal is construct MDEs satisfying certain properties.

4 Existence Theorem

There is always at least one solution to the *homogeneous* equation (3.3), namely $Y = 0$, the zero matrix. By applying standard methods from linear algebra to the coefficient matrix in equation (3.3), we can analyze how properties of the matrix X affect solutions to the MDE. In particular, we determine how many linearly independent solutions exist to the first-order MDE above.

Theorem 4.1. *For every $k \in \{0, 1, 2, 4\}$, there exists a first-order MDE over $M_2(\mathbb{C})$ whose solution space is k dimensional.*

Proof. We provide a specific example for each k in $\{0, 1, 2, 4\}$ of a first-order MDE over $M_2(\mathbb{C})$ having a k dimensional solution space. From the previous section, we saw how the solution dimension of

$$\Delta_X(Y) + aY = 0 \tag{4.1}$$

is related to the coefficient matrix

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a-x_{11}+x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a+x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix}.$$

The examples below were systematically generated by applying Gaussian elimination to this coefficient matrix and making choices for a and the x_{ij} that resulted in a desired number of pivots. To simplify the exposition, we simply list the resulting examples below. Our other theorem, a nonexistence result, guides the reader through our case-based, symbolic Gaussian elimination technique.

Case $k = 0$ (Unique solution): Consider $a = 1$ and $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Equation (3.3) becomes

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 4 & 0 & -2 \\ -3 & 0 & -2 & 3 \\ 0 & -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The uniqueness of the solution $Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ follows from the invertibility of the coefficient matrix, which we show by computing its determinant using cofactor expansion. Observe:

$$det \begin{pmatrix} 1 & 3 & -2 & 0 \\ 2 & 4 & 0 & -2 \\ -3 & 0 & -2 & 3 \\ 0 & -3 & 2 & 1 \end{pmatrix} = 1 \begin{pmatrix} 4 & 0 & -2 \\ 0 & -2 & 3 \\ -3 & 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 3 & -2 & 0 \\ 0 & -2 & 3 \\ -3 & 2 & 1 \end{pmatrix} - 3 \begin{pmatrix} 3 & -2 & 0 \\ 4 & 0 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= 4 \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix} - 3 \begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix} - 6 \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix} + 6 \begin{pmatrix} -2 & 0 \\ -2 & 3 \end{pmatrix}$$

$$-9 \cdot \begin{pmatrix} 0 & -2 \\ 2 & 1 \end{pmatrix} + 12 \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix} + 9 \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= (4)(-8) + (-3)(-4) + (-6)(-8) + (6)(-6) + (-9)(4) + (12)(-2) + (9)(4) = -32$$

As the determinant is nonzero, the coefficient matrix is invertible and the homogeneous system only has the zero solution.

Case $k = 1$ (One-dimensional solution space): Consider $a = 1$ and $X = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

The dimension of the solution space can be calculated by row-reducing the coefficient matrix and determining the number of non-pivot columns. We employ this method in the next two examples. The coefficient matrix in equation (3.3) is

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

To determine the nullity of the coefficient matrix on the left, we row-reduce

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three pivots, so the nullity equals 1. The system, and therefore the original MDE, has a one-dimensional solution space. By applying back-substitution to the row-echelon form, we find all solutions

$$\begin{aligned}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} &= \begin{pmatrix} y_{21} & -\frac{1}{2}y_{21} + \frac{1}{2}y_{22} \\ -y_{22} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} -y_{22} & -\frac{1}{2}(-y_{22}) + \frac{1}{2}y_{22} \\ -y_{22} & y_{22} \end{pmatrix} \\ &= y_{22} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\end{aligned}$$

Case $k = 2$ (Two-dimensional solution space): Consider $a = 0$ and $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. The coefficient matrix in equation (3.3) is

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

Row-reducing,

$$\begin{aligned}\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} &\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

There are two pivots, so the nullity equals $4 - 2 = 2$. The system has a two-dimensional solution space. The solutions are

$$\begin{aligned}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} &= \begin{pmatrix} y_{22} & y_{21} \\ y_{21} & y_{22} \end{pmatrix} \\ &= y_{21} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y_{22} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

Case $k = 4$ (Four-dimensional solution space): Consider $a = 0$ and $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. When $a = 0$, the MDE is $\Delta_X(Y) = 0$. Its solution space is the kernel of Δ_X. Since we are letting $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the derivation Δ_X is the zero map, so its kernel is all of $M_2(\mathbb{C})$. Therefore, the solution dimension is $\dim M_2(\mathbb{C}) = 4$. □

5 Nonexistence Theorem

We employ Gaussian elimination to our coefficient matrix for the first proof. As the matrix contains variables x_{ij} and a, there are various possibilities as to which entries can be pivots. An alternative proof is provided using eigenvalues and eigenvectors. These proofs provide different perspectives as well as distinct tools that may be used in extending this work to larger matrix algebras or higher order MDEs.

Theorem 5.1. *There does not exist a first-order, linear, constant-coefficient MDE over $M_2(\mathbb{C})$ having three linearly independent solutions.*

5.1 First Proof

Proof. We will use a proof by contradiction to show this matrix equation cannot have a three dimensional solution space. Assume the MDE

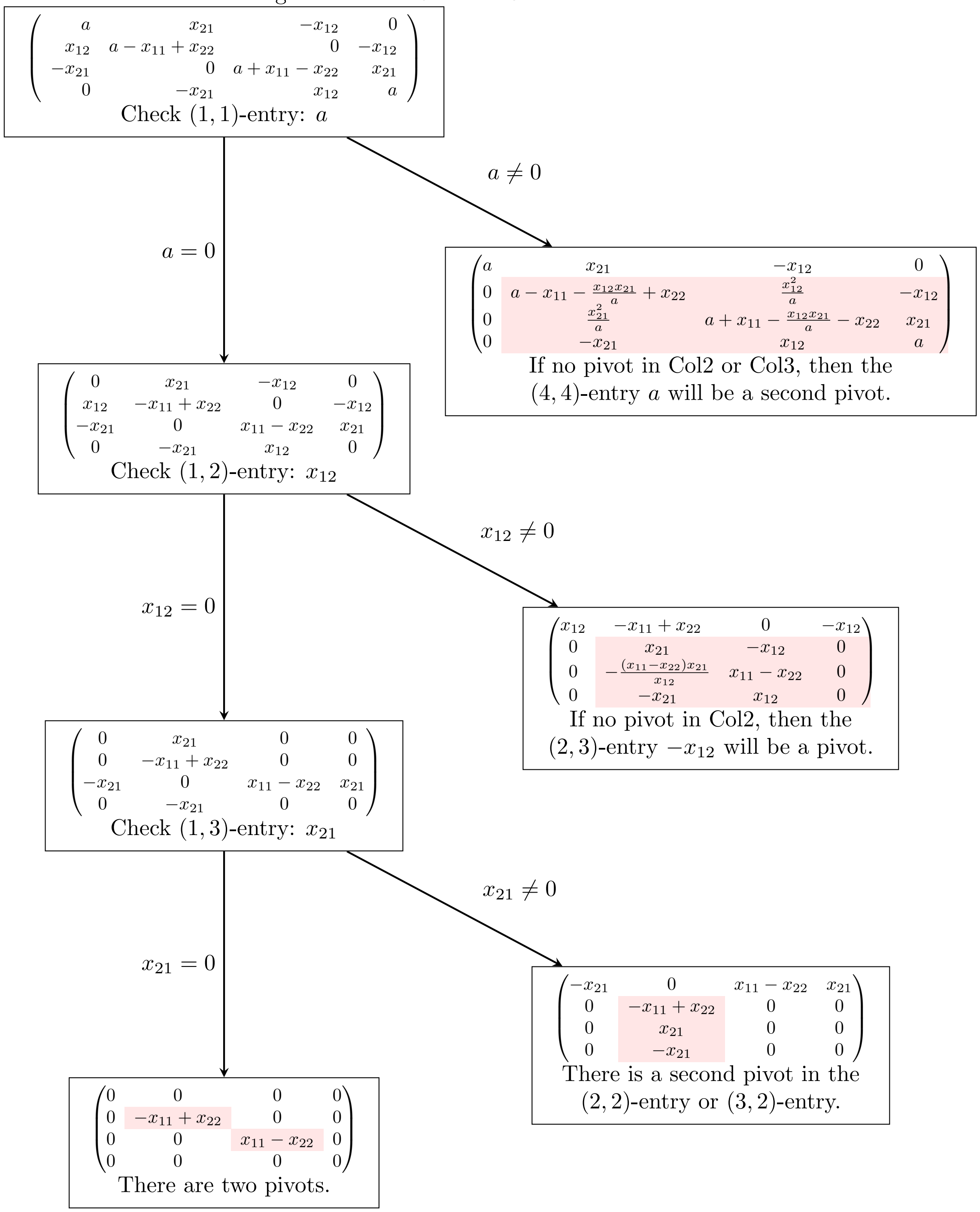
$$\Delta_X(Y) + aY = 0$$

has a three dimensional solution space. This is equivalent to the following coefficient matrix having nullity 3.

$$A = \begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix}.$$

By the Rank-Nullity Theorem, the rank of A must be 1. The rank of a matrix is also the number of pivots in its reduced row-echelon form. However, row-reduction requires we know whether various expressions involving a and the x_{ij} variables are zero or nonzero. Consequently, we will have nested cases based on whether certain entries can be pivots. To assist the reader in navigating the hierarchy of cases we encounter during Gaussian elimination, the following figure is helpful.

Figure 1: Pivot Cases in Gaussian Elimination.



We proceed with Gaussian elimination. First, consider the $(1,1)$-entry a. There are two cases: either the $(1,1)$-entry $a = 0$ or $a \neq 0$. The cases below will be labeled by sequences of L's and R's to reflect the corresponding navigation in the binary tree figure above.

Case L: We will begin the case when $a = 0$. This allows us to deduce that a is not a pivot. The matrix A now looks like

$$\begin{pmatrix} 0 & x_{21} & -x_{12} & 0 \\ x_{12} & -x_{11}+x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & 0 \end{pmatrix}$$

We then move on to the next position below and consider the $(2,1)$-entry x_{12}. We again have two cases based on whether x_{12} is zero or nonzero.

Case LL: We assume $x_{12}=0$. Therefore, x_{12} is not a pivot. We continue going down the column and consider the $(3,1)$-entry to be zero or nonzero.

Case LLL: We will assume the case for $-x_{21}$ to be zero, thus, completing the first column. We then move on to the next column and assign our unknown variables to be zero or nonzero. We can see that the $(1,2)$-entry x_{21} is zero from our previous assumption. Therefore, we move on to the next position below. We now have to consider the $(2,2)$-entry $-x_{11}+x_{22}$. The matrix A now looks like

$$\begin{pmatrix} 0 & 0 & 0 & \\ 0 & -x_{11}+x_{22} & 0 & 0 \\ 0 & 0 & x_{11}-x_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that after our previous declarations, we are left with two expressions that are the opposite of each other: $-x_{11}+x_{22}$ and $x_{11}-x_{22}$. By making both of the (2,2)-entry $-x_{11}+x_{22}$ and the (3,3)-entry $x_{11}-x_{22}$ equal zero, then that would give us the zero matrix, thus, having no pivots. If we assign both of the expressions to be nonzero, then that will give us two pivots. Either outcome contradicts the starting assumption of only having one pivot.

Case LLR: Now we can trace back and look at the cases where our unknown variables are nonzero. By going back a step, we can look at the case when x_{21} is nonzero. We have identified a pivot in Col 1, so we swap rows to get zeros below this pivot.

$$\begin{pmatrix} 0 & x_{21} & 0 & 0 \\ 0 & -x_{11}+x_{22} & 0 & 0 \\ -x_{21} & 0 & x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -x_{21} & 0 & x_{11}-x_{22} & x_{21} \\ 0 & -x_{11}+x_{22} & 0 & 0 \\ 0 & x_{21} & 0 & x_{21} \\ 0 & -x_{21} & 0 & 0 \end{pmatrix}$$

In order to have only one pivot, Gaussian elimination must encounter zeros in remaining entries. This cannot be achieved, since x_{21} is nonzero we find it later in the $(3,2)$-entry. Note, however, that this is more of a worst-case scenario. We may alternatively pick up an earlier pivot in case earlier expressions such as $-x_{11}+x_{22}$ are nonzero. Without having to explore various zero/nonzero subcases, the presence of the worst-case scenario contradicts the starting assumption of only having one pivot.

Case LR: We will now trace back to when we made an assumption about x_{12}, but now assume x_{12} to be nonzero. Then x_{12} will be our first pivot. We swap rows to get x_{12} in the $(1,1)$-entry.

$$\begin{pmatrix} 0 & x_{21} & -x_{12} & 0 \\ x_{12} & -x_{11}+x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & 0 \end{pmatrix} \sim \begin{pmatrix} x_{12} & -x_{11}+x_{22} & 0 & -x_{12} \\ 0 & x_{21} & -x_{12} & 0 \\ -x_{21} & 0 & x_{11}-x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & 0 \end{pmatrix}$$

As we scan through the rest of the row-swapped matrix, we can see that x_{12} appears in other entries such as the $(2,3)$-entry and the $(4,3)$-entry. We may pick up a second pivot before Gaussian elimination makes it that far, but the presence of those nonzero entries prevent us from having only one pivot.

Case R: We can now trace back to when we made our assumption on a and look at the case when a is nonzero. After some row operations,

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} \sim \begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ 0 & a - x_{11} - \frac{x_{12}x_{21}}{a} + x_{22} & \frac{x_{12}^2}{a} & -x_{12} \\ 0 & \frac{x_{21}^2}{a} & a + x_{11} - \frac{x_{12}x_{21}}{a} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix}.$$

As we scan through this new matrix, we notice the a in the far southeast corner. Gaussian elimination may find a second pivot (and possibly more) before getting to the $(4,4)$-entry. If not, then the $(4,4)$-entry will be another pivot. This contradicts our starting assumption of having only one pivot.

In all cases, we never have only one pivot. Therefore, the solution dimension of the the first-order MDE

$$\Delta_X(Y) + aY = 0$$

can never have solution dimension equal to three. □

5.2 Second Proof

We now provide an alternate proof of Theorem 5.1 using eigenvalues and eigenvectors. The main idea is to represent matrices X and Y with respect to a basis of eigenvectors or generalized eigenvectors for the matrix X. The matrix representation of the differential operator $\Delta_X + a$ has a more simple form than previously encountered, expressing the pivot possibilities in terms of a and eigenvalue differences. This method has a hint of combinatorics that is likely to play a much larger role if one considers MDEs over larger matrix algebras.

Alternate proof. First, we note that solution dimension is not affected by changing bases. Suppose $P \in M_2(\mathbb{C})$ is invertible and $\Psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is the conjugation map

$$\Psi(Y) = P^{-1}YP.$$

We compute

$$\begin{aligned} \Psi(\Delta_X(Y)) &= P^{-1}(YX - XY)P \\ &= P^{-1}YXP - P^{-1}XYP \\ &= P^{-1}YPP^{-1}XP - P^{-1}XPP^{-1}YP \\ &= \Psi(Y)\Psi(X) - \Psi(X)\Psi(Y) \\ &= \Delta_{\Psi(X)}(\Psi(Y)) \end{aligned}$$

Consequently,

$$\Psi(\Delta_X(Y) + aY) = \Delta_{\Psi(X)}(\Psi(Y)) + a\Psi(Y).$$

This equation tells us that Y is a solution to

$$\Delta_X(Y) + aY = 0$$

if and only if $\Psi(Y)$ is a solution to

$$\Delta_{\Psi(X)}(Z) + aZ = 0.$$

Since the solution spaces are in bijective correspondence via the *linear* isomorphism Ψ, the solution dimensions are the same.

We have two cases for X:

- X has two linearly independent eigenvectors v_1 and v_2.

- X has an eigenvector v_1 and a genearlized eigenvector v_2 that are linearly independent.

Case 1: Assume X has linearly eigenvectors v_1 and v_2 corresponding to eigenvalues λ_1 and λ_2. At this point, we make no assumption whether or not $\lambda_1 = \lambda_2$. Using v_1 and v_2 as a basis for $\mathbb{C}^2$, we may assume

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $Yv_1 = y_{11}v_1 + y_{21}v_2$ and $Yv_2 = y_{12}v_1 + y_{22}v_2$ so that we can write Y as the matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

We now have

$$\begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a - \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & 0 & a + \lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

If $a = 0$ and $\lambda_1 = \lambda_2$, then the matrix above has zero pivots. If $a = 0$, but the eigenvalues are distinct, then there are two pivots. If $a \neq 0$, then there are at least two pivots. Although not needed for this proof, $a \neq 0$ yields four pivots if $a \notin \{0, \lambda_1 - \lambda_2, \lambda_2 - \lambda_1\}$ and three pivots if $a \in \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_1\}$. Observe that we never obtained only one pivot.

Case 2: We now assume $\lambda_1 = \lambda_2$ and that X has the form

$$X = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

The differential operator $\Delta_X + aI$ now has matrix representation

$$\begin{aligned} \begin{pmatrix} a & x_{21} & -x_{12} & 0 \\ x_{12} & a - x_{11} + x_{22} & 0 & -x_{12} \\ -x_{21} & 0 & a + x_{11} - x_{22} & x_{21} \\ 0 & -x_{21} & x_{12} & a \end{pmatrix} &= \begin{pmatrix} a & 0 & -1 & 0 \\ 1 & a & 0 & -1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & a & 0 & -1 \\ a & 0 & -1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & a & 0 & -1 \\ 0 & -a^2 & -1 & a \\ 0 & 0 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & a & 0 & -1 \\ 0 & -a^2 & -1 & a \\ 0 & 0 & 1 & a \\ 0 & 0 & a & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & a & 0 & -1 \\ 0 & -a^2 & -1 & a \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \end{aligned}$$

In this case, we have two pivots if $a = 0$ and four pivots if $a \neq 0$. In no case did we ever have one pivot, so it is not possible for a homogeneous, first-order, costant-coefficient MDE over $M_2(\mathbb{C})$ to have three linearly independent solutions. □

Remark 5.2. In the case that X had a full basis of eigenvectors, both X and $\Delta_X + aI$ are diagonalizable. More generally for $M_n(\mathbb{C})$, the basis-of-eigenvectors case results in an $n^2 \times n^2$ diagonalizable matrix. One realization of this diagonal has n occurences of a and the rest of the $n^2 - n$ entries have the form $a \pm (\lambda_i - \lambda_j)$. Determining how many ways these entries could equal zero or nonzero is a combinatorial problem.

References

[BD69] W. E. Boyce and DiPrima. *Elementary differential equations and boundary value problems*. Volume 9. Wiley New York, 1969.

[GL14] L. Guo and F. Li. Structure of Hochschild cohomology of path algebras and differential formulation of Euler's polyhedron formula. *Asian j. math.*, 18(3):545–572, 2014.

[Hoc42] G. Hochschild. Semi-simple algebras and generalized derivations. *American journal of mathematics*, 64(1):677–694, 1942.

[Jac37] N. Jacobson. Abstract derivation and lie algebras. *Transactions of the american mathematical society*, 42(2):206–224, 1937.

[Noe33] E. Noether. Nichtkommutative algebra. *Mathematische zeitschrift*, 37(1):514–541, 1933.

[Rit32] J. F. Ritt. *Differential equations from the algebraic standpoint*. Volume 14. American Mathematical Soc., 1932.

[Wed08] J. Wedderburn. On hypercomplex numbers. *Proceedings of the london mathematical society*, 2(1):77–118, 1908.