COMPLETELY CONTRACTIVE EXTENSIONS OF HILBERT MODULES OVER TENSOR ALGEBRAS

ANDREW K. GREENE

Abstract. This paper studies completely contractive extensions of Hilbert modules over tensor algebras over C^*-correspondences. Using a result of Sz-Nagy and Foiaş on triangular contractions, extensions are parametrized in terms of contractive intertwining maps between certain defect spaces. These maps have a simple description when initial data consists of partial isometries. Sufficient conditions for the vanishing and nonvanishing of completely contractive Hilbert module Ext are given that parallel results for the classical disc algebra.

1. Introduction

This paper studies completely contractive extensions of Hilbert modules over tensor algebras over C^*-correspondences. These operator algebras were introduced by Muhly and Solel in [MS98]. Motivated by Douglas and Paulsen's pioneering work [DP89] on Hilbert modules over function algebras, Carlson and Clark introduced Ext theory for Hilbert modules over the disc algebra. Since then there have been numerous studies of extensions and their related derivations such as in [CCFW95; Clo14a; Clo14b; Fer96; Fer97]

Also in [MS98] Muhly and Solel paramertize completely contractive representations of tensor algebras in terms of certain intertwining contractions. The aim of this paper is provide a similar parametrization of $\mathrm{Ext}^1(K,H)$ for completely contractive Hilbert modules over $\mathscr{T}_+(X)$. An important tool in this analysis is a factorization result of Sz.-Nagy and Foiaş [SF67] that concerns contractions in triangular form. In some cases, it can determine when all derivations must be inner, or alternatively, suggest a construction for non-inner derivations. Related constructions have appeared in different contexts in [Dun07; Dun08; Pop96].

2. Preliminaries

We recall the construction of the tensor algebra over a C^*-correspondence E as well as Muhly and Solel's characterization of completely contractive representations.

Let E be a C^*-correspondence over a unital C^*-algebra A with left action $\phi : A \to \mathcal{L}(E)$. We let $E^{\otimes n} = E \otimes_A E \otimes_A \cdots \otimes_A E$ be the n-fold internal tensor product of E. The left A-action is given by the $*$-homomorphism $\phi_n : A \to \mathcal{L}(E^{\otimes n})$ satisfying

$$\phi_n(a)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

By convention we declare $E^{\otimes 0} = A$ as a C^*-correspondence over itself.

2010 *Mathematics Subject Classification.* Primary 46H25. Secondary 47L75 .

Definition 2.1. The *full Fock space* $\mathscr{F}(E)$ *over* E is the C^*-correspondence over A defined as $\oplus_{n=0}^{\infty} E^{\otimes n} = A \oplus E \oplus (E \otimes_A E) \oplus \cdots$. The left A-module structure is $\oplus_n \phi_n$ which we denote by ϕ_∞. It can be represented by the diagonal matrix

$$\phi_\infty(a) = \begin{bmatrix} a & & & \\ & \phi(a) & & \\ & & \phi_2(a) & \\ & & & \ddots \end{bmatrix}, \quad a \in A$$

where $\phi_n(a)(x_1 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes \cdots \otimes x_n$. Looking at the $(1,1)$-entry, it is clear that ϕ_∞ is injective. Thus, we will often identify A with its image $\phi_\infty(A)$.

For each $x \in E$, we define the creation operator $T_x \in \mathcal{L}(\mathscr{F}(E))$ by

$$T_x = \begin{bmatrix} 0 & & & & \\ T_x^{(1)} & 0 & & & \\ & T_x^{(2)} & 0 & & \\ & & T_x^{(3)} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

where $T_x^{(k)} : E^{\otimes k} \to E^{\otimes(k+1)}$ is given by the formula
$T_x^{(k)}(x_1 \otimes \cdots \otimes x_k) = x \otimes x_1 \otimes \cdots \otimes x_k$.

Definition 2.2. Let E be a C^*-correspondence over A.

(1) The *tensor algebra* of E, denoted $\mathscr{T}_+(E)$, is the norm closed subalgebra of $\mathcal{L}(\mathscr{F}(E))$ generated by $\phi_\infty(A)$ and $\{T_x \mid x \in E\}$.
(2) The *Toeplitz algebra* is the C^*-algebra generated by $\mathscr{T}_+(E)$ in $\mathcal{L}(\mathscr{F}(E))$.

Tensor algebras, introduced by Muhly and Solel in [MS98], are non-selfadjoint subalgebras of the Toeplitz C^*-algebras associated to E, which, in turn, were originally defined by Pimsner in [Pim97]. Tensor algebras have an attractively tractable completely contractive representation theory expressed in terms of maps defined on the C^*-correspondence E. The purely algebraic tensor algebras have an analogous property, cf. [Coh03].

If $\sigma_i : A \to B(H_i)$ are representations for $i = 1, 2$, then the *intertwining space* of σ_2 and σ_1, denoted $\mathcal{I}(\sigma_2, \sigma_1)$, is defined as the space of operators $T \in B(H_2, H_1)$ such that $T\sigma_2(a) = \sigma_1(a)T$ for every $a \in A$. In Theorem 3.10 and Lemma 3.5 of [MS98], the completely contractive representations of $\mathscr{T}_+(E)$ are parametrized in terms of certain contractive intertwiners between $E \otimes_\sigma H \to H$ Recall that $\sigma^E : \mathcal{L}(E) \to B(E \otimes_\sigma H)$ is the representation of $\sigma : A \to B(H)$ *induced up to* E and satisfies

$$\sigma^E(F)(x \otimes h) = F(x) \otimes h.$$

In fact, $\sigma^E(F) = F \otimes I_H$.

Definition 2.3. Let

$$E^\sigma = \mathcal{I}(\sigma^E \circ \phi, \sigma) = \{T \in B(E \otimes_\sigma H, H) \mid T(\phi(a) \otimes I_H) = \sigma(a)T \quad \forall a \in A\}$$

and let $\mathbb{D}(E^\sigma)$ be its open disc.

Then, the points in $\overline{\mathbb{D}(E^\sigma)}$ parametrize those completely contractive representations of $\mathscr{T}_+(E)$ *that equal* σ *when restricted to* $\phi_\infty(A)$. Of course, $\overline{\mathbb{D}(E^\sigma)}$, can also be written as $\big(\mathcal{I}(\sigma^E \circ \phi, \sigma)\big)_1$, but the disc notation emphasizes the perspective that elements of $\mathscr{T}_+(E)$ are functions on their space of representations. Indeed, when $E = A = \mathbb{C}$, $\mathscr{T}_+(\mathbb{C}) = \mathcal{A}(\mathbb{D})$ and $\mathbb{D}(E^\sigma) = \mathbb{D}$; the function-theoretic perspective is emphasized to the point of being part of the definition.

In [**CC95**] Carlson and Clark defined $\mathrm{Ext}^1(K, H)$ for Hilbert modules H and K over the disc algebra. This group is defined as a collection of equivalence classes of short exact sequences

$$0 \to H \xrightarrow{\alpha} J \xrightarrow{\beta} K \to 0$$

where H and K are prescribed Hilbert modules. Through similarity equivalences, J can be taken to be $H \oplus K$, which is the orthogonal direct sum of Hilbert spaces and not necessarily a module direct sum. In this way, the representation on $H \oplus K$ takes the form $\left(\begin{smallmatrix} \pi & \delta \\ 0 & \rho \end{smallmatrix}\right)$. It turns out that δ is a derivation and $\mathrm{Ext}^1(K, H)$ can be characterized as equivalence classes of derivations modulo inner derivations. In the case of the disc algebra, derivations are uniquely determined by the image of $f(z) = z$.

The perspective of this paper is to consider completely contractive extensions for more general tensor algebras $\mathscr{T}_+(E)$ over C^*-correspondences E where derivations are determined by maps $E \otimes_A K \to H$ satisfying certain intertwining conditions.

3. Extensions

We will need the following result of Sz.-Nagy and Foiaş [SF67]. It can be proven by using Douglas lemma [**Dou66**] and comparing the defect operator of the block operator matrix with the defects of S and T.

Proposition 3.1. *Suppose $S \in B(H_1, H_2)$ and $T \in B(K_1, K_2)$. Then $\|\left(\begin{smallmatrix} S & \Delta \\ 0 & T \end{smallmatrix}\right)\| \leq 1$ if and only if $\Delta = \sqrt{1 - SS^*} Y \sqrt{1 - T^*T}$ for some (unique) contraction $Y : \mathfrak{D}_T \to \mathfrak{D}_{S^*}$.*

This result was also used in [**CCFW95**] to study extensions of Hilbert modules over the disc algebra. To apply in our setting we set $H_1 = H_2 = H$ and $K_1 = K_2 = E \otimes_A K$. Then, $\mathfrak{D}_T \subseteq E \otimes_A K$ and $\mathfrak{D}_{S^*} \subseteq H$.

Proposition 3.2. *Completely contractive extensions of H_π by K_ρ are parametrized by contractions $Y : \mathfrak{D}_T \to \mathfrak{D}_{S^*}$ that intertwine $\sigma_1^E \circ \phi$ and σ_2.*

Proof. The first part of this result is a straightforward application of Proposition 3.1, However, the intertwining condition distinguishes this result from the classical disc algebra and noncommutative disc algebras where the intertwining condition is follows automatically from linearity.

First, we assume $H \oplus K$ is a Hilbert module is an extension of H by K corresponding to a completely contractive representation $\psi = \left(\begin{smallmatrix} \pi & \delta \\ 0 & \rho \end{smallmatrix}\right) : \mathscr{T}_+(E) \to H \widetilde{\oplus} K$. Denote $\sigma = \sigma_1 \oplus \sigma_2$. Then ψ corresponds to an element $X \in \overline{\mathbb{D}(E^\sigma)}$. The domain of X is $E \otimes_A (H \oplus K)$ which is the same as $(E \otimes_A H) \oplus (E \otimes_A K)$. With the respect to this decomposition, X has the form $\left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right)$. Observe that if $x \in E$ and $h \in H$, then the triangular form of ψ implies

$$X(x \otimes h) = \psi(T_x)h = \pi(T_x)h = S(x \otimes h).$$

Thus, $X_{11} = S$ and $X_{21} = 0$. Similarly, for $k \in K$,

$$X(x \otimes k) = \psi(T_x)k = \rho(T_x)k + \delta(T_x)k = T(x \otimes k) + X_{12}(x \otimes k).$$

Observe that T and X_12 have orthogonal ranges $E \otimes K$ and $E \otimes H$, respectively. Therefore, $X_22 = T$. Use Δ in place of X_{12}. Observe that $\Delta(x \otimes k) = \delta(T_x)k$. The derivation δ is encoded in the map $\Delta : E \otimes K \to H$. Observe that for $a \in A$,

$$\begin{aligned}\begin{pmatrix}\sigma_1(a)S & \sigma_1(a)\Delta \\ 0 & \sigma_2(a)T\end{pmatrix} &= \begin{pmatrix}\sigma_1(a) & 0 \\ 0 & \sigma_2(a)\end{pmatrix}\begin{pmatrix}S & \Delta \\ 0 & T\end{pmatrix} \\ &= \sigma(a)X \\ &= X\sigma^E(\phi(a) \\ &= X(\sigma_1^E(\phi(a) \oplus \sigma_2^E(\phi(a)) \\ &= \begin{pmatrix}S\sigma_1^E(\phi(a)) & \Delta\sigma_2^E(\phi(a)) \\ 0 & T\sigma_2^E(\phi(a))\end{pmatrix}\end{aligned}$$

Focusing on the $(1,2)$-entry we see that Δ intertwines $\sigma_2^E \circ \phi$ and σ_1. As $X = \left(\begin{smallmatrix} S & \Delta \\ 0 & T \end{smallmatrix}\right)$ is a triangular contraction, there exists some contraction $Y : \mathfrak{D}_T \to \mathfrak{D}_{S^*}$. Since $S \in \overline{\mathbb{D}(E^{\sigma_1})}$ and $T \in \overline{\mathbb{D}(E^{\sigma_2})}$, it follows from their intertwining conditions that SS^* commutes with $\sigma_1(a)$ and T^*T commutes with $\sigma_2^E(\phi(a))$ for every $a \in A$. Consequently, $\sqrt{1 - SS^*}$ commutes with $\sigma_1(A)$ and $\sqrt{1 - T^*T}$ commutes with $\sigma_2^E(A)$. Therefore, for $a \in A$,

$$D_{S^*}\sigma_1(a)YD_T = \sigma_1(a)\Delta = \Delta\sigma_2^E(\phi(a)) = D_{S^*}Y\sigma_2^E(\phi(a))D_T.$$

Considered as operators from $\mathfrak{D}_T$ to $\mathfrak{D}_{S^*}$, we have $\sigma_1(a)Y = Y\sigma_2^E(\phi(a))$, as desired.

Conversely, given such a contraction Y, settting $\Delta := D_{S^*}YD_T$ results in $X = \left(\begin{smallmatrix} S & \Delta \\ 0 & T \end{smallmatrix}\right)$ being contractive. If Y intertwines $\sigma_2^E \circ \phi$ and σ_1, then so does Δ. The matrix computations above (after suitable rearranging) show that X then intertwines $\sigma^E \circ \phi$ with σ. Therefore, X yields a completely contractive representation ψ on $H \oplus K$. The triangular form of X implies a triangular form of ψ which yields a completely contractive extension of H by K. □

4. Equivalent Extensions

Two extensions are equivalent if their corresponding derivations differ by an inner derivation. To understand the previous result from a cohomology perspective, one needs to know when two $X_1, X_2 \in \overline{\mathbb{D}(E^\sigma)}$ are equivalent. Because the notation $\overline{\mathbb{D}(E^\sigma)}$ refers to intertwining maps from $E \otimes (H \oplus K) \to H \oplus K$, but the parameters Y are contractive intertwiners from $\mathfrak{D}_T$ to $\mathfrak{D}_{S^*}$, we introduce the notation

$$\overline{\mathbb{D}(T, S^*)} := \{Y : \mathfrak{D}_T \to \mathfrak{D}_{S^*} \mid \|Y\| \leq 1 \text{ and } \sigma_1(a)Y = Y\sigma_2^E(\phi(a)) \quad \forall a \in A\}.$$

Proposition 4.1. *Given $Y_1, Y_2 \in \overline{\mathbb{D}(T, S^*)}$, their corresponding extensions are equivalent if and only if there exists $X \in B(K, H)$ such that X intertwines σ_2 and σ_1 and*

$$\sqrt{1 - SS^*}(Y_2 - Y_1)\sqrt{1 - T^*T} = S(1_E \otimes X) - XT.$$

Proof. Let δ_1 and δ_2 be the corresponding derivations to Y_1 and Y_2. The equation follows from the form of an inner derivation: Given $x \in E$ and $k \in K$,

$$\sqrt{1 - SS^*}(Y_2 - Y_1)\sqrt{1 - T^*T}(x \otimes k) = (\Delta_2 - \Delta_1)(x \otimes k) = (\delta_2 - \delta_1)(T_x)k \tag{4.1}$$

On the other hand,

$$S(1_E \otimes X) - XT)(x \otimes k) = S(x \otimes Xk) - XT(x \otimes k) = (\pi(T_x)X - X\rho(T_x))k \tag{4.2}$$

The expressions in equation 4.1 equal the expressions in equation 4.2 for arbitrary $x \in E$ and $k \in K$ if and only if $\delta_2 - \delta_1$ is an inner derivation.

Since we only consider completely contractive Hilbert modules, then the representations $\left(\begin{smallmatrix} \pi & \delta_i \\ 0 & \rho \end{smallmatrix}\right)$ are C^*-representations when restricted to A. Consequently, $\delta_i(A) = 0$ for $i = 1, 2$. Note that by the product rule, this also says that the δ_i are linear over A. Consequently, the inner derivation implemented by $X \in B(K, H)$ vanishes on A if and only if X intertwines σ_2 and σ_1. □

Remark 4.2. In analogy to derivations, we will say Δ is inner if $\Delta = S(1 \otimes X) - XT$ for some X.

If σ_1 and σ_2 are disjoint C^*-representations, then there do not exist nonzero inner derivations. Consequently, each completely contractive extension represents a distinct cohmology class in $\mathrm{Ext}^1(K, H)$.

More generally, if σ_1 and σ_2 are disjoint and $x \in E$ is an element with $\phi(a)x = xa$ for all $a \in A$, then $\delta(T_x) = 0$ for any derivation vanishing on A.

Paulsen proved in [Pau84] that an operator algebra homomorphism is completely bounded if and only if it is similar to a complete contraction. The extension theory presently under consideration is invariant under similarity, so propositions 3.2 and 4.1 imply

Theorem 4.3. *In the category of completely bounded Hilbert modules, $Ext^1(K, H)$ is equivalent to the collection of all equivalence classes of $Y \in \overline{\mathbb{D}(S, T^*)}$ under the equivalence $Y_1 \sim Y_2$ if*

$$\sqrt{1 - SS^*}(Y_2 - Y_1)\sqrt{1 - T^*T} = S(1_E \otimes X) - XT$$

for some $X \in B(K, H)$ intertwining σ_2 and σ_1.

5. Partial Isometries

In this section we study extensions when S and T are partial isometries. In this case, the defect operators are projections and we examine the factorization in proposition 3.2 with respect to their corresponding subspace decompsitions. We denote the null space of T by $N(T)$ and the range of S by $R(S)$.

Theorem 5.1. *Suppose S and T are partial isometries. Then completely contractive Hilbert module extensions are parametrized by contractions*

$$Y : N(T) \to R(S)^\perp$$

that intertwine $\sigma_2^E \circ \phi$ and σ_1. Moreover, such Y correspond to noninner derivations if and only if $Y \neq 0$.

Proof. For partial isometries S and T, we have projections SS^* onto $R(S)$ the (closed) range of S and T^*T onto $N(T)^\perp$, the orthocomplement of the null space of T. Therefore, $D_{S^*} = Proj(R(T)^\perp)$ and $D_T = Proj(N(T))$. It follows from proposition 3.2 that completely contractive extensions are parametrized by contractions

$$Y : N(T) \to R(S)^\perp.$$

This also means Δ is supported on $N(T)$ and has range contained in $R(S)$. If we decompose $E \otimes_A K = N(T) \oplus N(T)^\perp$ and $H = R(S) \oplus R(S)^\perp$, then

$$\Delta = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}.$$

Nonzero maps Y are automatically noninner because $S(1 \otimes X) - XT$ maps $N(T)$ into $R(S)$ for any $X \in B(K, H)$. As a matrix,

$$S(1 \otimes X) - XT = \begin{pmatrix} SX(1 - T^*T) & S(1 \otimes X)T^*T - SS^*XT \\ 0 & (1 - SS^*XT) \end{pmatrix}$$

By looking at the two matrices above, we conclude Δ is inner if and only if $\Delta = 0$. □

Corollary 5.2. *If S is a coisometry or T is an isometry, then $\Delta = 0$.*

Proof. This is actually corollary of the proof that does not require both S and T to be partial isometries. If S is a cosimetry, then $D_{S^*} = 0$, which makes Y, and therefore Δ, equal to 0. If T is an isometry, then $D_T = 0$ which makes Y and Δ automatically 0. □

Remark 5.3. The above corollary is well-known. It says that isometries are projective Hilbert modules in the completely bounded category.

6. Non-inner Derivations

The previous section reveals an easy way to produce nontrivial extensions when S and T are partial isometries, provided S is not surjective and T is not injective. Such examples have appeared in [Dun07; Dun08; Pop96], especially in the particular situation in which $S = T = 0$ and $N(T)$ and $R(S)^\perp$ are as large as possible.

Although, the assumptions of S and T being partial isometries resulted in simple expressions for the defects, they are unnecessary.

Example 6.1. Assume T and S^* have nontrivial null spaces such that there exist nonzero vectors $v \in N(T) \subseteq E \otimes_A K$ and $w \in N(S^*) \subseteq H$ (which we may take to have norm one) satisfying

$$\langle \phi(a)\eta \otimes k, v\rangle w = \sigma_1(a)\left(\langle \eta \otimes k, v\rangle w\right) \quad \forall \eta \in E, \quad \forall k \in K.$$

Note that this condition is equivalent to the rank one operator $Y = w \otimes \bar{v}$ intertwining $\sigma_2^E \circ \phi$ and σ_1. Let $Y = w \otimes \overline{v}$. We claim that $\Delta = D_{S^*} Y D_T = Y$. First, observe that $(1 - SS^*)w = w$. Since D_{S^*} can by uniformly approximated by polynomials in $1 - SS^*$ and having constant term 1, then $D_{S^*} w = w$ as well. Similarly, $D_T v = v$. Consequently,

$$\Delta = D_{S^*}(w \otimes \bar{v}) D_T = (D_{S^*} w) \otimes \left(\overline{D_{T^*}^* v}\right) = w \otimes \bar{v}.$$

Observe that

$$\Delta v = \langle v, v\rangle w = \|v\|^2 w = 1w = w.$$

This means that Δ cannot possibly be inner. For any $X \in B(K, H)$,

$$(S(1 \otimes X) - XT)v = S(1 \otimes X)v - 0 \in R(S)$$

and, therefore, cannot equal $w \notin R(S)$.

The intertwining condition is trivially satisfied by linearity in case $A = \mathbb{C}$. It would be interesting to understand this conditions in common settings directed graph tensor algebras, analytic crossed products, and semicrossed products. In [**Pop96**] Popescu uses a different argument to determine that $\mathrm{Ext}^1(\mathbb{C}, \mathbb{C}) = \mathbb{C}^d$ when $H = K = \mathbb{C}$ with $S = T = 0$. In that case, the representation is interpreted as "evaluation at 0." Such "point derivations" were studied in higher dimensions and for different directed graphs by Duncan in [**Dun07**; **Dun08.**]

References

[CCFW95] J. F. Carlson, D. N. Clark, C. Foias, and J. P. Williams. Projective Hilbert **A**(**D**)-modules. *New york j. math.*, 1:26–38, electronic, 1994/95.

[Clo14a] R. Clouâtre. On the unilateral shift as a Hilbert module over the disc algebra. *Complex anal. oper. theory*, 8(1):283–309, 2014.

[Clo14b] R. Clouâtre. Spectral and homological properties of Hilbert modules over the disc algebra. *Studia math.*, 222(3):263–282, 2014.

[Coh03] P. M. Cohn. *Further algebra and applications.* Springer-Verlag London Ltd., London, 2003, pages xii+451.

[DP89] R. G. Douglas and V. I. Paulsen. *Hilbert modules over function algebras.* Volume 217 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow, 1989, pages vi+130.

[Dun07] B. L. Duncan. Derivations for a class of matrix function algebras. *Linear algebra appl.*, 422(1):67–76, 2007.

[Dun08] B. L. Duncan. Finite dimensional point derivations for graph algebras. *Illinois j. math.*, 52(2):419–435, 2008.

[Fer96] S. H. Ferguson. Polynomially bounded operators and Ext groups. *Proc. amer. math. soc.*, 124(9):2779–2785, 1996.

[Fer97] S. H. Ferguson. Backward shift invariant operator ranges. *J. funct. anal.*, 150(2):526–543, 1997.

[MS98] P. S. Muhly and B. Solel. Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes. *J. funct. anal.*, 158(2):389–457, 1998.

[Pau84] V. I. Paulsen. Completely bounded homomorphisms of operator algebras. *Proc. amer. math. soc.*, 92(2):225–228, 1984.

[Pim97] M. V. Pimsner. A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by **Z**. In, *Free probability theory (Waterloo, ON, 1995).* Volume 12, in Fields Inst. Commun. Pages 189–212. Amer. Math. Soc., Providence, RI, 1997.

[Pop96] G. Popescu. Non-commutative disc algebras and their representations. *Proc. amer. math. soc.*, 124(7):2137–2148, 1996.

[SF67] B. Sz.- and C. Foiaş. Forme triangulaire d'une contraction et factorisation de la fonction caractéristique. *Acta sci. math. (szeged)*, 28:201–212, 1967.

Department of Mathematics, Manhattan College, Riverdale, NY 10471
E-mail address: `andrew.greene@manhattan.edu`