Matrix Differential Equations: Noncommutative Variation of Parameters

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Outline of topics

Motivational Example

Consider the following second-order, linear, constant coefficient, nonhomogeneous ODE:

$$y'' + ay' + by = g.$$

From ODEs to MDEs

1. Replace functions y and g with matrices Y and G.

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2. Replace $\frac{d}{dx}$ with a derivation $\Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R})$

$$\frac{d}{dx} \to \Delta.$$

Given $a, b \in \mathbb{R}$, $G \in M_n(\mathbb{R})$, and Δ, solve for Y satisfying

$$\Delta^2(Y) + a\Delta(Y) + bY = G.$$

Derivations

Definition

A derivation $\Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is

- a linear transformation

$$\Delta(a_1 Y_1 + a_2 Y_2) = a_1 \Delta(Y_1) + a_2 \Delta(Y_2) \text{ for all } Y_1, Y_2 \in M_n(\mathbb{R})$$

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Example

Fix $X \in M_n(\mathbb{R})$ and define $\Delta_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by

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Example (the only example for $M_n(\mathbb{R})$)

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Solving MDEs

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$$\begin{aligned}\Delta^2(Y) + a\Delta(Y) + bY &= G \\ \Delta(YX - XY) + a(YX - XY) + bY &= G\end{aligned}$$

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$$\begin{aligned}
\Delta^2(Y) + a\Delta(Y) + bY &= G \\
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\end{aligned}$$

Solving MDEs

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$$A \begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} g_{11} \\ g_{12} \\ g_{21} \\ g_{22} \end{pmatrix}$$

where A equals

$$\begin{pmatrix} 2x_{12}x_{21} + b & ax_{21} - (x_{11} - x_{22})x_{21} & -ax_{12} - (x_{11} - x_{22})x_{12} & -2x_{12}x_{21} \\ ax_{12} - x_{11}x_{12} + x_{12}x_{22} & -a(x_{11} - x_{22}) + (x_{11} - x_{22})x_{11} + 2x_{12}x_{21} - (x_{11} - x_{22})x_{22} + b & -2x_{12}^2 & -ax_{12} + x_{11}x_{12} - x_{12}x_{22} \\ -ax_{21} - x_{11}x_{21} + x_{21}x_{22} & -2x_{21}^2 & a(x_{11} - x_{22}) + (x_{11} - x_{22})x_{11} + 2x_{12}x_{21} - (x_{11} - x_{22})x_{22} + b & ax_{21} + x_{11}x_{21} - x_{21}x_{22} \\ -2x_{12}x_{21} & -ax_{21} + (x_{11} - x_{22})x_{21} & ax_{12} + (x_{11} - x_{22})x_{12} & 2x_{12}x_{21} + b \end{pmatrix}$$

Solution Dimension

Theorem [Persaud, 2015]

For each $k \in \{0, 1, 2, 4\}$ there exists a 1st order MDE over $M_2(\mathbb{R})$ of the form

$$\Delta_X(Y) + aY = 0$$

whose space of solutions $\mathcal{S}$ has dimension k.

Nonexistence Theorem [Garcia, G., & Persaud, 2016]

There does not exist a 1st order, constant coefficient, linear MDE over $M_2(\mathbb{R})$ whose solution space has dimension 3.

(Kevin Garcia - Matrix Differential Equations and Gaussian Elmination, Friday, 10:10 AM, MAA Session #13, Madison Room)

Variation of Parameters

Suppose $Y_1, Y_2, \ldots, Y_k$ are a basis of solutions to the homogeneous MDE

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Consider a solution to the nonhomogeneous MDE

$$\Delta^2(Y) + a\Delta(Y) + bY = G$$

of the form

$$Y = \sum_{i=1}^{k} Y_i U_i.$$

Variation of Parameters

Substitute

$$\Delta(Y_i U_i) = \Delta(Y_i)U_i + Y_i\Delta(U_i).$$

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$$\Delta^2(Y_i U_i) = \Delta^2(Y_i)U_i + 2\Delta(Y_i)\Delta(U_i) + Y_i\Delta^2(U_i).$$

Variation of Parameters

$$\begin{aligned} G &= \Delta^2(Y) + a\Delta(Y) + bY \\ &= \sum_{i=1}^{k} \Delta^2(Y_i)U_i + 2\Delta(Y_i)\Delta(U_i) + Y_i\Delta^2(U_i) \\ &\qquad + a\Delta(Y_i)U_i + aY_i\Delta(U_i) + bY_iU_i \end{aligned}$$

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&= \left(\sum_{i=1}^{k} \Delta(Y_i)V_i\right) + \Delta\left(\sum_{i=1}^{k} Y_iV_i\right) + a\left(\sum_{i=1}^{k} Y_iV_i\right)
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- Assume $\sum_{i=1}^{k} Y_i V_i = 0$.

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- Assume $\sum_{i=1}^{k} Y_iV_i = 0$.
- It follows that $\sum_{i=1}^{k} \Delta(Y_i)V_i = G$.

Wronskian

$$\begin{pmatrix} Y_1 & Y_2 & \cdots & Y_{k-1} & Y_k \\ \Delta(Y_1) & \Delta(Y_2) & \cdots & \Delta(Y_{k-1}) & \Delta(Y_k) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{k-1} \\ V_k \end{pmatrix} = \begin{pmatrix} 0 \\ G \end{pmatrix}$$

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ODEs

- Classicaly, the Wronskian $W(Y_1, \ldots, Y_k) = \left(\Delta^{i-1}(Y_j)\right)$ is a square matrix.

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- W is often used to test whether the Y_i are linearly independent.

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- W is often used to test whether the Y_i are linearly independent.
- Cramer's rule is used to write V_i as ratios of determinants.

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MDEs:

- W no longer needs to be a square matrix.

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MDEs:

- W no longer needs to be a square matrix.
- Trivial null space $\implies$ $Y_1, \ldots Y_k$ linearly independent.
- Full rank $\implies$ the above equation is solvable for arbitrary G.

"Integration"

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- Even if when solutions $V_1, \ldots, V_k$ exist, we still require U_i satisfying $V_i = \Delta(U_i)$.

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- Contrast: basic integration $\int 1\, dx = x + C$ in the classical case.

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- The range of Δ consists only of matrices with trace-zero.
- Consequently, $I_n = \Delta(U_i) = U_i X - X U_i$ has no solution.
- Contrast: basic integration $\int 1\,dx = x + C$ in the classical case.
- In the matricial case, linear algebra can be used to study the range of Δ and solve "integrals." (Solve $\Delta(U_i) = V_i$ for U_i).

Thank you

Thank you very much for your attention.

Upcoming Talk

Matrix Differential Equations and Gaussian Elimination
Kevin Garcia
Friday, Aug 5, 2016 10:10 - 10:25 AM
MAA Session #13, Madison Room